

# Weak and strong approximations of reflected diffusions via penalization methods

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## Abstract

We study approximations of reflected Itô diffusions on convex subsets  $D$  of  $\mathbb{R}^d$  by solutions of stochastic differential equations with penalization terms. We assume that the diffusion coefficients are merely measurable (possibly discontinuous) functions. In the case of Lipschitz continuous coefficients we give the rate of  $\mathbb{L}^p$  approximation for every  $p \geq 1$ . We prove that if  $D$  is a convex polyhedron then the rate is  $\mathcal{O}((\frac{\ln n}{n})^{1/2})$ , and in the general case the rate is  $\mathcal{O}((\frac{\ln n}{n})^{1/4})$ .

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## 1 Introduction

In the paper we study weak and strong approximations of solutions of  $d$ -dimensional stochastic differential equations (SDEs)

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds + K_t, \quad t \in \mathbb{R}^+ \quad (1.1)$$

with reflecting boundary condition on a convex domain  $D$ . Here  $x_0 \in \bar{D} = D \cup \partial D$ ,  $X$  is a reflecting process on  $\bar{D}$ ,  $K$  is a bounded variation process with variation  $|K|$  increasing only, when  $X_t \in \partial D$ ,  $W$  is a  $d$ -dimensional standard Wiener process and  $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ ,  $b : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  are measurable (possibly discontinuous) functions. Suppose that for  $n \in \mathbb{N}$  we are given measurable coefficients  $\sigma_n : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ ,  $b_n : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and a standard Wiener process  $W^n$ , and assume that there exists a solution  $X^n$  of the following SDE with penalization term

$$X_t^n = x_0 + \int_0^t \sigma_n(s, X_s^n) dW_s^n + \int_0^t b_n(s, X_s^n) ds - n \int_0^t (X_s^n - \Pi(X_s^n)) ds, \quad t \in \mathbb{R}^+, \quad (1.2)$$

where  $\Pi(x)$  is the projection of  $x$  on  $\bar{D}$ . The problem is to find conditions on  $\{\sigma_n\}$ ,  $\{b_n\}$  ensuring convergence of  $\{X^n\}$  to the reflected diffusion  $X$ , and secondly, to give the rate of such convergence.

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Reflected diffusions have many applications, for instance in queueing systems, seismic reliability analysis and finance (see e.g. Asmussen [3], Dupuis and Ramanan [6], Krée and Soize [10], Pettersson [18], Shepp and Shiryaev [24]). Therefore, the problem of practical approximations of solutions of (1.1) is very important. Discrete penalization schemes based on the approximation of  $X$  by solutions of equations with penalization term are well known (see e.g. Pettersson [19], Kanagawa and Saisho [8], Liu [13], Słomiński [26]).

Approximation of reflected diffusions via penalization methods was earlier considered by Menaldi [16], Menaldi and Robin [17], Lions and Sznitman [11], Lions, Menaldi and Sznitman [12], Storm [22], Saisho and Tanaka [22] and many others. Unfortunately, these authors have restricted themselves to the case of Lipschitz continuous coefficients.

In the present paper we consider measurable coefficients  $\sigma_n, b_n$  such that

$$\|\sigma_n(t, x)\|^2 + |b_n(t, x)|^2 \leq C(1 + |x|^2), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \quad n \in \mathbb{N} \quad (1.3)$$

for some  $C > 0$ . To prove convergence of  $\{X^n\}$  to  $X$  we first show that under (1.3) the sequence  $\{X^n\}$  is very close to the sequence  $\{\Pi(X^n)\}$  and we observe that  $\Pi(X^n)$  is a solution of some Skorokhod problem (for the definition of the Skorokhod problem see Section 2). Next, using a well developed theory of convergence of solutions of the Skorokhod problem (see e.g. [3, 20, 25, 26, 29]) we prove our main approximation results. Moreover, we are able to strengthen the rate of the convergence of the penalization method in the classical case of Lipschitz continuous coefficients  $\sigma, b$ .

The paper is organized as follows.

In Section 2 we estimate the  $\mathbb{L}^p$  distance between  $X^n$  and  $\bar{D}$ . Using some new estimates of  $\mathbb{L}^p$ -modulus of continuity of Itô's processes from Fischer and Nappo [7] we prove that under (1.3) for every  $p \geq 1$ ,  $T > 0$ ,

$$\|\sup_{t \leq T} \text{dist}(X_t^n, \bar{D})\|_p = \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^{1/2}\right),$$

where  $\|\cdot\|_p = (E(\cdot)^p)^{1/p}$  denotes the usual  $\mathbb{L}^p$  norm. We also show that  $\{X^n\}$  is tight in  $C(\mathbb{R}^+, \mathbb{R}^d)$  and its weak limit point solve the Skorokhod problem.

Section 3 contains our main results concerning weak and strong approximations of solutions of (1.1). We consider the set of conditions on coefficients from the paper by Rozkosz and Słomiński [20] on stability of solutions of stochastic differential equations with reflecting boundary. Roughly speaking, we assume that  $\{\sigma_n\}, \{b_n\}$  satisfy (1.3) and  $\{(\det \sigma_n \sigma_n^*)^{-1}\}$  is locally uniformly integrable on some set ( $\sigma_n^*$  denotes the matrix adjoint to  $\sigma_n$ ). Then we show that if  $\{\sigma_n\}, \{b_n\}$  tend to  $\sigma, b$  a.e. on the set mentioned above and uniformly on its completion, then  $X^n \rightarrow_{\mathcal{P}} X$ , where  $X$  denotes a unique weak solution of (1.1). Under the additional assumptions that  $W^n \rightarrow_{\mathcal{P}} W$  and that (1.1) is pathwise unique we show that  $X^n \rightarrow_{\mathcal{P}} X$ . Thus, we generalize earlier approximation results to equations with possibly discontinuous and nonelliptic diffusion coefficients and discontinuous drift coefficients.

Section 4 is devoted to the classical case, where all coefficients are fixed Lipschitz continuous functions with respect to  $x$  and all stochastic integrals are driven by the

same Wiener process, i.e.  $\sigma_n = \sigma$ ,  $b_n = b$ , and  $W_n = W$ ,  $n \in \mathbb{N}$ , and there is  $L > 0$  such that

$$\|\sigma(t, x) - \sigma(t, y)\|^2 + |b(t, x) - b(t, y)|^2 \leq L|x - y|^2, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d. \quad (1.4)$$

In this case, if  $D$  is a convex polyhedron, we prove that for every  $p \geq 1$ ,  $T > 0$ ,

$$\|\sup_{t \leq T} |X_t^n - X_t|\|_p = \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^{1/2}\right).$$

For arbitrary convex domain we prove that for every  $p \geq 1$ ,  $T > 0$ ,

$$\|\sup_{t \leq T} |X_t^n - X_t|\|_p = \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^{1/4}\right).$$

Thus, we strengthen earlier results on the subject proved by Menaldi [16].

In the sequel we use the following notation.  $\mathbb{R}^+ = [0, \infty)$ ,  $C(\mathbb{R}^+, \mathbb{R}^d)$  is the space of continuous functions  $x : \mathbb{R}^+ \rightarrow \mathbb{R}^d$  equipped with the topology of uniform convergence on compact subsets of  $\mathbb{R}^+$ . For every  $x \in C(\mathbb{R}^+, \mathbb{R}^d)$ ,  $\delta > 0$ ,  $T > 0$  we set  $\omega_\delta(x, T) = \sup\{|x_t - x_s|; s, t \in [0, T], |s - t| \leq \delta\}$ .  $\mathbb{R}^d \otimes \mathbb{R}^d$  is the set of  $(d \times d)$ -matrices. The abbreviation *a.e.* means “almost everywhere” with respect to the Lebesgue measure, “ $\rightarrow_{\mathcal{D}}$ ”, “ $\rightarrow_{\mathcal{P}}$ ” denote convergence in law and in probability, respectively.

## 2 General results

Let  $D$  be a nonempty convex domain in  $\mathbb{R}^d$  and let  $\mathcal{N}_x$  denote the set of inward normal unit vectors at  $x \in \partial D$ . Note that  $\mathbf{n} \in \mathcal{N}_x$  if and only if  $\langle y - x, \mathbf{n} \rangle \geq 0$  for every  $y \in \bar{D}$  (see e.g. [16, 22]). Moreover, if  $\text{dist}(x, \bar{D}) > 0$ , then

$$\frac{\Pi(x) - x}{|\Pi(x) - x|} \in \mathcal{N}_{\Pi(x)}.$$

Let  $Y$  be an  $\{\mathcal{F}_t\}$ -adapted process with continuous trajectories. We will say that a pair  $(X, K)$  of  $\{\mathcal{F}_t\}$ -adapted processes is a solution of the Skorokhod problem associated with  $Y$  if

$$X = Y + K,$$

$$X \text{ is } \bar{D}\text{-valued,}$$

$$K \text{ is a process with locally bounded variation such that } K_0 = 0 \text{ and}$$

$$K_t = \int_0^t \mathbf{n}_s d|K|_s, \quad |K|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} d|K|_s, \quad t \in \mathbb{R}^+,$$

$$\text{where } \mathbf{n}_s \in \mathcal{N}_{X_s} \text{ if } X_s \in \partial D.$$

It is well known that for every process  $Y$  with continuous trajectories there exists a unique solution  $(X, K)$  of the Skorokhod problem associated with  $Y$  (see e.g. [4] or [15], where a more general case of càdlàg processes is considered). The theory of convergence of solutions of the Skorokhod problem is well developed (see e.g. [3, 20, 25, 26, 29]).

Unfortunately, solutions of (1.2) are not solutions of the Skorokhod problem and the problem of their convergence is more delicate.

Suppose that we are given a filtered probability space  $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}, P^n)$  satisfying the usual conditions and a  $d$ -dimensional  $\{\mathcal{F}_t^n\}$ -Wiener process  $W^n$ ,  $n \in \mathbb{N}$ . Let  $\{X^n\}$  denote the sequence of solutions of (1.2). In the present paper we will use the simple fact that under (1.3) there exists a sequence of solutions of the Skorokhod problem very close to the sequence  $\{X^n\}$ . Observe that we can rewrite (1.2) into the form

$$\Pi(X_t^n) = Y_t^n - n \int_0^t (X_s^n - \Pi(X_s^n)) ds, \quad t \in \mathbb{R}^+,$$

where  $Y_t^n = x_0 - X_t^n + \Pi(X_t^n) + \int_0^t \sigma_n(s, X_s^n) dW_s^n + \int_0^t b_n(s, X_s^n) ds$ ,  $t \in \mathbb{R}^+$ ,  $n \in \mathbb{N}$ . Since  $\Pi(X^n) \in \bar{D}$ ,  $|K^n|$  increases only when  $\Pi(X^n)_t \in \partial D$  and

$$K_t^n = n \int_0^t \frac{\Pi(X_s^n) - X_s^n}{|\Pi(X_s^n) - X_s^n|} |\Pi(X_s^n) - X_s^n| ds = \int_0^t \mathbf{n}_s d|K^n|_s, \quad t \in \mathbb{R}^+,$$

it is clear that  $(\Pi(X^n), K^n)$  is a solution of the Skorokhod problem associated with  $Y^n$ ,  $n \in \mathbb{N}$ . One can also observe that

$$|X_t^n - \Pi(X_t^n)| = \text{dist}(X_t^n, \bar{D}), \quad t \in \mathbb{R}^+, n \in \mathbb{N}.$$

**Theorem 2.1** *Assume that (1.3) is satisfied.*

(i) *For every  $p \geq 1$ ,  $T > 0$  there is  $C > 0$  such that*

$$\|\sup_{t \leq T} \text{dist}(X_t^n, \bar{D})\|_p \leq C \left( \frac{\ln n}{n} \right)^{1/2}, \quad n \in \mathbb{N}.$$

(ii)  *$\{(X^n, K^n)\}_{n \in \mathbb{N}}$  is tight in  $C(\mathbb{R}^+, \mathbb{R}^{2d})$  and its every weak limit point  $(X, K)$  is a solution of the Skorokhod problem.*

PROOF. (i) Fix  $T > 0$ . First observe that by [14, Corollary 2.4] and Gronwall's lemma,

$$\sup_n E \sup_{t \leq T} |X_t^n|^p < +\infty \quad (2.1)$$

for every  $p \geq 1$ . By the above and estimates from Fischer and Nappo [7, Theorem 1] (see also [18, Lemma 4.4] and [26, Lemma A4]) for every  $p \geq 1$  there is  $C > 0$  such that

$$\|\omega_{1/n}(\bar{Y}^n, T)\|_p \leq C \left( \frac{\ln n}{n} \right)^{1/2}, \quad n \in \mathbb{N}, \quad (2.2)$$

where  $\bar{Y}_t^n = x_0 + \int_0^t \sigma_n(s, X_s^n) dW_s^n + \int_0^t b_n(s, X_s^n) ds$ ,  $t \in \mathbb{R}^+$ ,  $n \in \mathbb{N}$ .

Fix  $n \in \mathbb{N}$  and  $k = 0, 1, \dots, [nT] - 1$ . Clearly,  $X^n$  is a solution of the equation

$$X_{k/n+s}^n = X_{k/n}^n + \bar{Y}_{k/n+s}^n - \bar{Y}_{k/n}^n - n \int_0^s (X_{k/n+u}^n - \Pi(X_{k/n+u}^n)) du, \quad s \in [0, 1/n] \quad (2.3)$$

on the interval  $[k/n, (k+1)/n]$ . It is also clear that there exists a unique solution of the equation

$$\bar{X}_s^n = X_{k/n}^n - n \int_0^s (\bar{X}_u^n - \Pi(\bar{X}_u^n)) du, \quad s \in [0, 1/n]. \quad (2.4)$$

One can easily check that  $\bar{X}_s^n = \Pi(X_{k/n}^n) + (X_{k/n}^n - \Pi(X_{k/n}^n))e^{-ns}$ ,  $s \in [0, 1/n]$ , which implies that

$$|\bar{X}_{1/n}^n - \Pi(\bar{X}_{1/n}^n)| \leq |\bar{X}_{1/n}^n - \Pi(X_{k/n}^n)| = |X_{k/n}^n - \Pi(X_{k/n}^n)|e^{-1}. \quad (2.5)$$

Subtracting (2.4) from (2.3) we see that

$$X_{k/n+s}^n - \bar{X}_s^n = \bar{Y}_{k/n+s}^n - \bar{Y}_{k/n}^n - n \int_0^s (X_{k/n+u}^n - \bar{X}_u^n - \Pi(X_{k/n+u}^n) + \Pi(\bar{X}_u^n)) du, \quad s \in [0, 1/n],$$

hence that

$$|X_{k/n+s}^n - \bar{X}_s^n| \leq \omega_{1/n}(\bar{Y}^n, T) + 2n \int_0^s |X_{k/n+u}^n - \bar{X}_u^n| du, \quad s \in [0, 1/n],$$

because  $\Pi : \mathbb{R}^d \rightarrow \bar{D}$  is Lipschitz continuous with the constant equal to 1. Applying Gronwall's lemma we conclude from the above that

$$|X_{(k+1)/n}^n - \bar{X}_{1/n}^n| \leq e^2 \omega_{1/n}(\bar{Y}^n, T). \quad (2.6)$$

Setting  $A_k^n = |X_{k/n}^n - \Pi(X_{k/n}^n)|$  and using (2.5), (2.6) we have

$$\begin{aligned} A_{k+1}^n &\leq |X_{(k+1)/n}^n - \Pi(\bar{X}_{1/n}^n)| \leq |X_{(k+1)/n}^n - \bar{X}_{1/n}^n| + |\bar{X}_{1/n}^n - \Pi(\bar{X}_{1/n}^n)| \\ &\leq e^2 \omega_{1/n}(\bar{Y}^n, T) + e^{-1} A_k^n. \end{aligned}$$

Since  $A_0^n = 0$  and  $A_1^n \leq e^2 \omega_{1/n}(\bar{Y}^n, T)$ , by induction on  $k$  we obtain

$$\max_{0 \leq k \leq [nT]} |X_{k/n}^n - \Pi(X_{k/n}^n)| \leq \frac{e^2}{1 - e^{-1}} \omega_{1/n}(\bar{Y}^n, T). \quad (2.7)$$

Furthermore, for  $k = 0, 1, \dots, [nT]$  and  $s \in [0, 1/n]$  such that  $k/n + s \leq T$ ,

$$\begin{aligned} X_{k/n+s}^n - X_{k/n}^n &= \bar{Y}_{k/n+s}^n - \bar{Y}_{k/n}^n \\ &\quad - n \int_0^s ((X_{k/n+u}^n - X_{k/n}^n) - (\Pi(X_{k/n+u}^n) - \Pi(X_{k/n}^n))) du \\ &\quad - ns(X_{k/n}^n - \Pi(X_{k/n}^n)). \end{aligned}$$

Hence, by Gronwall's lemma,

$$\sup_{s \in [0, 1/n]} |X_{(k/n+s) \wedge T}^n - X_{k/n}^n| \leq e^2 (\omega_{1/n}(\bar{Y}^n, T) + |X_{k/n}^n - \Pi(X_{k/n}^n)|),$$

which when combined with (2.7) gives

$$\max_{0 \leq k \leq [nT]} \sup_{s \in [0, 1/n]} |X_{(k/n+s) \wedge T}^n - X_{k/n}^n| \leq C \omega_{1/n}(\bar{Y}^n, T) \quad (2.8)$$

for some  $C > 0$ . Of course (2.7), (2.8) and (2.2) imply (i).

(ii) By (1.3), (2.1) and the well known Aldous criterion (see e.g. [2]),

$$\{\bar{Y}^n\} \text{ is tight in } C(\mathbb{R}^+, \mathbb{R}^d).$$

Moreover,  $\{\bar{Y}^n\}$  satisfies the so called UT condition (see e.g. [25, 26]) and hence its every weak limit point is a semimartingale. Due to part (i), the sequence  $\{Y^n\}$  is also tight in  $C(\mathbb{R}^+, \mathbb{R}^d)$ . Assume that  $Y^{(n)} \xrightarrow{\mathcal{D}} \bar{Y}$  in  $C(\mathbb{R}^+, \mathbb{R}^d)$  along some subsequence. By [26, Corollary A3],

$$(X^{(n)}, K^{(n)}) \xrightarrow{\mathcal{D}} (X, K) \text{ in } C(\mathbb{R}^+, \mathbb{R}^{2d}),$$

where  $(\bar{X}, \bar{K})$  is a unique solution of the Skorokhod problem associated with a semimartingale  $\bar{Y}$ .  $\square$

**Remark 2.2** Under (1.3),

$$\sup_n E|K^n|_T^p < +\infty \quad (2.9)$$

for every  $p \geq 1$ ,  $T > 0$ . This follows from (2.1) and [14, Theorem 2.5].

### 3 Approximations of weak and strong solutions

We say that the SDE (1.1) has a strong solution if there exists a pair  $(X, K)$  of  $\{\mathcal{F}_t\}$ -adapted processes satisfying (1.1) and such that  $(X, K)$  is a solution of the Skorokhod problem associated with

$$Y_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds, \quad t \in \mathbb{R}^+. \quad (3.1)$$

Recall also that the SDE (1.1) is said to have a weak solution if there exists a probability space  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ , an  $\{\bar{\mathcal{F}}_t\}$ -adapted Wiener process  $\bar{W}$  and a pair of  $\{\bar{\mathcal{F}}_t\}$ -adapted processes  $(\bar{X}, \bar{K})$  satisfying (1.1) with  $\bar{W}$  instead of  $W$ .

The following set of general conditions was introduced in Rozkosz and Słomiński [20]. We say that condition (H) is satisfied if for some closed subsets  $H, H_1$  of  $\mathbb{R}^+ \times \mathbb{R}^d$  such that  $H_1 \subset H$ ,

- $\forall \varepsilon > 0 \{(\det \sigma_n \sigma_n^*)^{-1}\}_{n \in \mathbb{N}}$  is uniformly integrable on each bounded subset of  $H^c(\varepsilon)$ ,
- $\sigma_n \rightarrow \sigma, b_n \rightarrow b$  a.e. on  $H^c = \mathbb{R}^+ \times \mathbb{R}^d \setminus H$ ,
- for every  $(t, x) \in H_1$  (for every  $(t, x) \in H$ ),

$$\sigma_n(t, x_n) \rightarrow \sigma(t, x), \quad b_n(t, x_n) \rightarrow b(t, x)$$

for all  $\{x_n\}$  such that  $x_n \rightarrow x$  (for all  $\{(t, x_n)\} \subset H$  such that  $x_n \rightarrow x$ ).

Here  $H^c(\varepsilon) = \mathbb{R}^+ \times \mathbb{R}^d \setminus H(\varepsilon)$  and  $H(\varepsilon) = H \cup H_{1,\varepsilon}$ , where  $H_{1,\varepsilon} = \emptyset$  if  $H_1 = \emptyset$  and  $H_{1,\varepsilon} = \{z \in \mathbb{R}^+ \times \mathbb{R}^d : \inf_{y \in H_1} |z - y| \leq \varepsilon\}$ , otherwise.

**Theorem 3.1** *Assume that (1.3) and (H) are satisfied.*

(i) *If the SDE (1.1) has a unique weak solution  $X$  then*

$$X^n \xrightarrow{\mathcal{D}} X \quad \text{in } C(\mathbb{R}^+, \mathbb{R}^d).$$

(ii) *If  $W^n \xrightarrow{\mathcal{P}} W$  in  $C(\mathbb{R}^+, \mathbb{R}^d)$  and the SDE (1.1) is pathwise unique then*

$$X^n \xrightarrow{\mathcal{P}} X \quad \text{in } C(\mathbb{R}^+, \mathbb{R}^d),$$

*where  $X$  is a unique strong solution of (1.1).*

PROOF. We use notations from the proof of Theorem 2.1.

(i) Our method of proof will be adaptation of the proof of [20, Theorem 2.2]. Since  $K^n$  is a bounded variation process, one can observe that Krylov's inequality used in [20, Theorem 5.1] is still in force, i.e. there exists a constant  $C$  depending only on  $d, R$  and  $t$  such that for every non-negative measurable  $f : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ ,

$$E \int_0^{t \wedge \tau_n^R} f(s, X_s^n) ds \leq C \|(\det \sigma_n \sigma_n^*)^{-1/(d+1)} f\|_{\mathbb{L}_{d+1}([0, t] \times B(0, R))}, \quad (3.2)$$

where  $\tau_n^R = \inf\{t : |X_t| \vee |K^n|_t > R\}$ ,  $B(0, R) = \{y \in \mathbb{R}^d : |y| < R\}$ . By Theorem 2.1(ii),  $\{(X^n, W^n)\}$  is tight in  $C(\mathbb{R}^+, \mathbb{R}^{2d})$  and we may assume that  $(X^{(n)}, W^{(n)}) \xrightarrow{\mathcal{D}} (\bar{X}, \bar{W})$  in  $C(\mathbb{R}^+, \mathbb{R}^{2d})$  along some subsequence, where  $\bar{W}$  is a Wiener process with respect to the natural filtration  $\mathcal{F}^{\bar{X}, \bar{W}}$ . By (3.2) and arguments from the proof of [20, Theorem 2.2],

$$(X^{(n)}, \bar{Y}^{(n)}, W^{(n)}) \xrightarrow{\mathcal{D}} (\bar{X}, \bar{Y}, \bar{W}) \quad \text{in } C(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where  $\bar{Y}_t = x_0 + \int_0^t \sigma(s, \bar{X}_s) d\bar{W}_s + \int_0^t b(s, \bar{X}_s) ds$ ,  $t \in \mathbb{R}^+$ . Since  $Y^n = \bar{Y}^n - X^n + \Pi(X^n)$ , it follows from Theorem 2.1(ii) that

$$(X^{(n)}, K^{(n)}, \bar{Y}^{(n)}) \xrightarrow{\mathcal{D}} (\bar{X}, \bar{Y}, \bar{K}) \quad \text{in } C(\mathbb{R}^+, \mathbb{R}^{3d}),$$

where  $(\bar{X}, \bar{K})$  is a solution of the Skorokhod problem associated with  $\bar{Y}$ . Hence  $(\bar{X}, \bar{K})$  is a weak solution of (1.1) and the result follows due to weak uniqueness of (1.1).

(ii) By using arguments from Gyöngy and Krylov [9], to prove that  $\{X^n\}$  converges in probability it is sufficient to show that from any subsequences  $(l) \subset (n), (m) \subset (n)$  it is possible to choose further subsequences  $(l_k) \subset (l), (m_k) \subset (m)$  such that  $(X^{(l_k)}, X^{(m_k)}) \xrightarrow{\mathcal{D}} (\bar{X}, \bar{X})$  in  $C(\mathbb{R}^+, \mathbb{R}^{2d})$ , where  $\bar{X}$  is a process with continuous trajectories. From Theorem 2.1(ii) we deduce that

$$\{(X^{(l)}, \bar{X}^{(m)}, W^{(l)}, W^{(m)})\} \quad \text{is tight in } C(\mathbb{R}^+, \mathbb{R}^d).$$

Therefore, we can choose subsequences  $(l_k) \subset (l), (m_k) \subset (m)$  such that

$$(X^{(l_k)}, X^{(m_k)}, W^{(l_k)}, W^{(m_k)}) \xrightarrow{\mathcal{D}} (\bar{X}', \bar{X}'', \bar{W}, \bar{W}), \quad \text{in } C(\mathbb{R}^+, \mathbb{R}^{4d}),$$

where  $\bar{X}', \bar{X}''$  are processes with continuous trajectories and  $\bar{W}$  is a Wiener process with respect to the natural filtration  $\mathcal{F}^{\bar{X}', \bar{X}'', \bar{W}}$ . In view of part (i), the processes  $\bar{X}', \bar{X}''$  are solutions of (1.1) with  $\bar{W}$  in place of  $W$ . Since (1.1) is pathwise unique,  $\bar{X}' = \bar{X}''$ , and consequently  $\{X^n\}$  converges in probability in  $C(\mathbb{R}^+, \mathbb{R}^{4d})$  to some continuous process  $X$ . Hence  $(X^n, W^n) \rightarrow_{\mathcal{P}} (X, W)$ , so using once again the pathwise uniqueness property of (1.1) shows that  $X$  is a unique strong solution of (1.1).  $\square$

**Remark 3.2** From [20] it follows that in fact in part (i) of the above theorem the assumption that  $\sigma_n \rightarrow \sigma$  a.e. on  $H^c$  may be replaced by a weaker assumption that  $\sigma_n \sigma_n^* \rightarrow \sigma \sigma^*$  a.e. on  $H^c$ .

**Remark 3.3** There are important examples of equations of the form (1.1) with discontinuous coefficients having unique weak or strong solutions. For instance, in Schmidt [21] it is shown that if  $d = 1$ ,  $D = (r_1, r_2)$ ,  $b \equiv 0$  and  $\sigma$  is purely function of  $x$ , then (1.1) has a unique weak solution for every starting point  $x_0 \in \bar{D}$  if and only if the set  $M$  of all  $x \in \bar{D}$  such that  $\int_{\bar{D} \cap U_x} \sigma^{-2}(y) dy = +\infty$  for every open neighborhood  $U_x$  of  $x$  is equal to the set  $N$  of zeros of  $\sigma$ . In multidimensional case it is known, that a solution of (1.1) is unique in law if (1.3) is satisfied with  $\sigma_n, b_n$  replaced by  $\sigma, b$ , the coefficient  $\sigma \sigma^*$  is bounded, continuous and uniformly elliptic, and  $\partial D$  is regular (see Stroock and Varadhan [28] for more details). Recently Semrau [23] considered the classical case  $d = 1$ ,  $D = \mathbb{R}^+$  with coefficients  $\sigma, b$  depending only on  $x$ . She has shown that if  $\sigma, b$ , satisfy (1.3),  $\sigma$  is uniformly positive and  $(\sigma(x) - \sigma(y))^2 \leq |f(x) - f(y)|$ ,  $x, y \in \mathbb{R}^+$  for some bounded increasing function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  then there exists a unique strong solution of (1.1). Some weaker results on pathwise uniqueness can be found in the earlier paper by Zhang [30].

Since in condition (H) we do not require continuity of the limit coefficients  $\sigma$  and  $b$ , Theorem 3.1 is a useful tool for practical approximations of solutions of the equations mentioned above.

#### 4 Rate of convergence in the case of Lipschitz continuous coefficients

In this section we assume that  $\sigma_n = \sigma$ ,  $b_n = b$ ,  $n \in \mathbb{N}$ , where  $\sigma, b$  are Lipschitz continuous functions with respect to  $x$ , i.e. satisfy (1.4). We also assume that all SDEs with penalization term are driven by a fixed  $\{\mathcal{F}_t\}$ -Wiener process  $W$ . In particular, this means that  $X^n$  is a solution of the equation

$$X_t^n = x_0 + \int_0^t \sigma(s, X_s^n) dW_s + \int_0^t b(s, X_s^n) ds - n \int_0^t (X_s^n - \Pi(X_s^n)) ds, \quad t \in \mathbb{R}^+. \quad (4.1)$$

Tanaka [29] has shown that in the case of Lipschitz continuous coefficients there exists a unique strong solution  $(X, K)$  of (1.1). Moreover, from [26, Theorem 2.2] and Gronwall's lemma it follows that

$$E \sup_{t \leq T} |X_t|^p < +\infty \quad \text{and} \quad E |K|_T^p < +\infty \quad (4.2)$$

for every  $p \geq 1$ ,  $T > 0$ .



**Theorem 4.1** Assume that (1.3) and (1.4) are satisfied. Let  $X^n$  satisfy (4.1),  $n \in \mathbb{N}$ . For every  $p \in \mathbb{N}$ ,  $T > 0$  there is  $C > 0$  such that

(i) if  $D$  is a convex polyhedron then

$$\|\sup_{t \leq T} |X_t^n - X_t|\|_p \leq C \left( \frac{\ln n}{n} \right)^{1/2}, \quad n \in \mathbb{N},$$

(ii) if  $D$  is a general convex domain then

$$\|\sup_{t \leq T} |X_t^n - X_t|\|_p \leq C \left( \frac{\ln n}{n} \right)^{1/4}, \quad n \in \mathbb{N},$$

where  $X$  is a unique strong solution of (1.1)

PROOF. Fix  $T > 0$ . Without loss of generality we may assume that  $p \geq 2$ .

(i) By Theorem 2.2 from Dupuis and Ishi [5] there exists  $C > 0$  such that

$$\begin{aligned} \sup_{s \leq t} |\Pi(X_s^n) - X_s| &\leq C \sup_{s \leq t} |Y_s^n - Y_s| \\ &\leq C \left( \sup_{s \leq t} |\Pi(X_s^n) - X_s^n| + \sup_{s \leq t} |\bar{Y}_s^n - Y_s| \right) \end{aligned} \quad (4.3)$$

for every  $t \leq T$ , where  $\bar{Y}_s^n = x_0 + \int_0^s \sigma(u, X_u^n) dW_u + \int_0^s b(u, X_u^n) du$ ,  $Y_s^n = \bar{Y}_s^n + \Pi(X_s^n) - X_s^n$ ,  $s \leq T$ ,  $n \in \mathbb{N}$ . Therefore, by Theorem 2.1(i), Burkholder-Davis-Gundy and Schwarz's inequalities,

$$\begin{aligned} E \sup_{s \leq t} |X_s^n - X_s|^p &\leq \text{Const} \left( \left( \frac{\ln n}{n} \right)^{p/2} + E \sup_{s \leq t} |\bar{Y}_s^n - Y_s|^p \right) \\ &\leq \text{Const} \left( \left( \frac{\ln n}{n} \right)^{p/2} + E \int_0^t \|\sigma(s, X_s^n) - \sigma(s, X_s)\|^p ds \right. \\ &\quad \left. + E \int_0^t |b(s, X_s^n) - b(s, X_s)|^p ds \right) \end{aligned}$$

for every  $t \leq T$ . By the above and (1.4),

$$E \sup_{s \leq t} |X_s^n - X_s|^p \leq \text{Const} \left( \left( \frac{\ln n}{n} \right)^{p/2} + \int_0^t E \sup_{u \leq s} |X_u^n - X_u|^p ds \right)$$

for every  $t \leq T$ , so (i) follows by Gronwall's lemma.

(ii) If  $D$  is a general convex domain then by Lemma 2.2 in Tanaka [29],

$$\begin{aligned} |\Pi(X_t^n) - X_t|^2 &\leq |\bar{Y}_t^n - Y_t|^2 + 2 \int_0^t (Y_t^n - Y_t - Y_s^n + Y_s) d(K_s^n - K_s) \\ &\leq \text{Const} (|\Pi(X_t^n) - X_t^n|^2 + |\bar{Y}_t^n - Y_t|^2 + \sup_{t \leq T} |\Pi(X_t^n) - X_t^n| (K^n|_T + |K|_T) \\ &\quad + \int_0^t (\bar{Y}_t^n - Y_t - \bar{Y}_s^n + Y_s) d(K_s^n - K_s) | \end{aligned} \quad (4.4)$$

for every  $t \leq T$ . Since by the integration by parts formula,

$$\begin{aligned} & \int_0^t (\bar{Y}_t^n - Y_t - \bar{Y}_s^n + Y_s) d(K_s^n - K_s) \\ &= \int_0^t (X_s^n - X_s) d(\bar{Y}_s^n - Y_s) + \frac{1}{2}([\bar{Y}^n - Y]_t - |\bar{Y}_t^n - Y_t|^2) \end{aligned}$$

(here  $[\bar{Y}^n - Y]$  denotes the quadratic variation of  $\bar{Y}^n - Y$ ), it follows from Theorem 2.1(i) and (4.4) that

$$\begin{aligned} E \sup_{s \leq t} |X_s^n - X_s|^p &\leq \text{Const}((\frac{\ln n}{n})^{p/2} + E \sup_{s \leq t} |\bar{Y}_s^n - Y_s|^p + E([\bar{Y}^n - Y]_t)^{p/2} \\ &\quad + E(\sup_{t \leq T} |\Pi(X_t^n) - X_t^n|)^{p/2} (|K^n|_t + |K|_t)^{p/2} \\ &\quad + E \sup_{s \leq t} \left| \int_0^t (X_s^n - X_s) d(\bar{Y}_s^n - Y_s) \right|^{p/2}. \end{aligned}$$

By Schwarz's inequality, Theorem 2.1(i), (2.9) and (4.2),

$$E(\sup_{t \leq T} |\Pi(X_t^n) - X_t^n|)^{p/2} (|K^n|_T + |K|_T)^{p/2} \leq \text{Const}((\frac{\ln n}{n})^{p/4}).$$

Since  $Y^n - Y$  is a continuous semimartingale with a martingale part  $M^n = \int_0^\cdot \sigma(s, X_s^n) - \sigma(s, X_s) dW_s$  and a bounded variation part  $V^n = \int_0^\cdot b(s, X_s^n) - b(s, X_s) ds$ , using Burkholder-Davis-Gundy and Schwarz's inequalities we get

$$\begin{aligned} E \sup_{s \leq t} \left| \int_0^t (X_s^n - X_s) d(\bar{Y}_s^n - Y_s) \right|^{p/2} &\leq \text{Const}(E(\int_0^t |X_s^n - X_s|^2 d[M^n]_s)^{p/4} \\ &\quad + E(\sup_{s \leq t} |X_s^n - X_s| |V^n|_t)^{p/2}) \\ &\leq \text{Const}(E \sup_{s \leq t} |X_s^n - X_s|^p)^{1/2} (E([M^n]_t)^{p/2} + (|V^n|_t)^p)^{1/2}. \end{aligned}$$

Observing that  $[\bar{Y}^n - Y] = [M^n]$  and using the elementary inequality  $2ab \leq \epsilon^2 a^2 + (b/\epsilon)^2$  with some sufficiently small  $\epsilon$  we deduce from the above that

$$\begin{aligned} E \sup_{s \leq t} |X_s^n - X_s|^p &\leq \text{Const}((\frac{\ln n}{n})^{p/4} + E \sup_{s \leq t} |\bar{Y}_s^n - Y_s|^p \\ &\quad + E([M^n]_t)^{p/2} + E(|V^n|_t)^p) \\ &\leq \text{Const}((\frac{\ln n}{n})^{p/4} + E \int_0^t \|\sigma(s, X_s^n) - \sigma(s, X_s)\|^p ds \\ &\quad + E \int_0^t |b(s, X_s^n) - b(s, X_s)|^p ds) \\ &\leq \text{Const}((\frac{\ln n}{n})^{p/4} + \int_0^t E \sup_{u \leq s} |X_u^n - X_u|^p ds) \end{aligned}$$

for every  $t \leq T$ . Using Gronwall's lemma completes the proof.  $\square$

**Remark 4.2** In the case of bounded convex domains and bounded Lipschitz continuous coefficients  $\sigma, b$  the problem of  $\mathbb{L}^p$  approximation of solutions of (1.1) by sequences of solutions of (4.1) was considered earlier in Menaldi [16]. In particular, in [16, Theorem 3.1] it is proved that for every  $p \geq 1$  and  $T > 0$ ,  $\|\sup_{t \leq T} |X_t^n - X_t|\|_p \rightarrow 0$ . From the proof of [16, Theorem 3.1] one can also deduce that

$$\forall_{\delta > 0} \|\sup_{t \leq T} |X_t^n - X_t|\|_p = \mathcal{O}\left(\left(\frac{1}{n}\right)^{1/4-\delta}\right). \quad (4.5)$$

In fact, in [16, Remark 3.1] a better rate is stated. However, R. Pettersson has observed that there is a gap in the proof of [16, Theorem 3.1] (in the first line on page 741  $p$  should be replaced by  $2p$ ). Using Menaldi's calculations and taking into account Pettersson's remark one can only prove (4.5). It is also worth pointing out that Menaldi's method of proof of (4.5) is completely different from our method based on estimates of  $\mathbb{L}^p$ -modulus of continuity for Itô processes.

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